The old and new Concepts of Physics
An open-dialogue journal

www.uni.lodz.pl/concepts

ON AN EXPERIMENTAL VERIFICATION OF RELATIVISTIC AND NONRELATIVISTIC PREDICTIONS FOR ELECTRON ENERGY LEVELS IN A STATIC MAGNETIC FIELD

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(Received 7 July 2008; accepted 7 July 2008)

Abstract

Physical consequences of the relativistic and nonrelativistic approaches to describe the energy levels of electrons which propagate in a static homogeneous magnetic field are considered. It is shown that for a given strength of the magnetic field, the quantized energy levels of the electrons calculated by nonrelativistic and relativistic equations differ substantially, up to few orders of magnitude for a magnetic field of about 1 Tesla. Experimental verification to resolve the discrepancy would be very welcome.

1 Introduction

The existence of quantized transverse energy levels of charged particles which propagate in a static homogeneous magnetic field was predicted by the nonrelativistic Schrödinger equation [1] and by the relativistic Klein-Gordon equation for a scalar particle [2, 3]. Later, analogous expressions for transverse energy levels of electrons were found within the Dirac equation (see [4]). For more details we refer to [5, 6, 7, 8]. Furthermore, exact solutions were also derived for somewhat more complicated cases when the electron pass through a combined static electric and magnetic field (Volkov type solutions, see also [9, 10]), and recently, this study was continued by finding the solution of the Dirac equation for a superposition of a static homogeneous magnetic and electric field including the anomalous magnetic moment of the electron [11]. It was also shown in ref.[11] that accounting for the electron anomalous magnetic moment the spin degeneracy of energy levels was removed. So, according to all these results, an electron in a static magnetic field gyrates about and stream along the field lines and possesses quasi-atomic bound states with energy levels related to its gyrating motion in the plane normal to the velocity vector. In the literature, these quasi-discrete resonance states were discussed in connection with the motion of charged particles in an external magnetic field and accelerator physics, see [12, 13, 14, 15] for example.

The analytical solutions of the corresponding equations allow to notice what kind of physical effects may appear during the transition from the nonrelativistic to the relativistic formalism. The example of the Coulomb potential, which we mention in what follows, shows a nontrivial physical sequence when passing from one case to the other.

In this note we would like to point the attention to the fact that the nonrelativistic and relativistic approaches, based on the Schrödinger, respectively, Dirac equation, give different analytical expressions for transverse energy levels of electrons in a static magnetic field. These solutions predict different dependencies of the transverse energy levels on the magnetic field, so that for a given field strength, rather different values for transverse energies are expected. In order to check the validity of the different theoretical predictions it would be very welcome to measure the energy of photons emitted by electrons when transitions from higher to lower orbits occur in an external magnetic field.

In Section 2 we discuss some basic formulas which define the energy levels of an electron which traverses a static and uniform magnetic field. Section 3 presents the numerical values for transverse energy levels of electrons assuming a relatively strong magnetic field of 1 Tesla. Section 4 summarizes the discussion and proposes an experimental verification to decide which of the two theoretical concepts based on either the Dirac or Schrödinger equation is realized.

Throughout this paper, the Gaussian system of units will be used.

2 Energy levels and wave function of electrons in a static magnetic field; solutions of the Schrödinger and Dirac equations

We consider a case of a static homogeneous magnetic field ($\vec{H} = rot\vec{A}$) with the following choice, according to ref.[1], of the 4-vector of the electromagnetic potential A_μ: $A_0 = A_x = A_z = 0$, $A_y = Hx$. Thus, $\vec{A} = Hx\vec{e_y}$, with $\vec{e_y}$ the unit vector along the y-axis, and $\vec{H} = (0, 0, H_z)$, with $H_z = H$.

The energy spectrum obtained from the Schrödinger equation for electrons with spin $\frac{1}{2}$, which moves parallel to the field lines of a static homogeneous magnetic field, i.e. along the z-axis, may be written as the sum of the longitudinal and transverse components [16]

$$E^{nonrel} = E_z^{nonrel} + E_{T,\lambda}^{nonrel}(n) \; , \tag{1}$$

where

$$E_z^{nonrel} = \frac{p_z^2}{2m_e} \tag{2}$$

and

$$E_{T,\lambda}^{nonrel}(n) = \hbar(\frac{eH}{2m_e c})(2n + 1 + 2\lambda) = (\mu_B^e H)(2n + 1 + 2\lambda) \; , \tag{3}$$

the energy of the electron motion in transverse direction [17]. The transverse energy depends on the strength of the magnetic field H and on the electron spin projection λ onto the z- (i.e. electron beam) direction. It possesses quantized values labeled by the main quantum number n ($n = 0, 1, 2, ...$). In eq.(3), m_e and e denote the electron mass, respectively, its charge, and $\frac{eH}{m_e c} = \omega_c$ is the cyclotron frequency. We employed here the definition of the Bohr magneton of an electron, $\mu_B^e = \frac{e\hbar}{2m_e c}$.

The energy levels, defined by the relativistic Dirac equation for electrons in the same static magnetic field [4], are connected to the fourth component of its 4-momentum vector, $P_\mu = (p^0, p^x, p^y, p^z)$, and are given as

$$c^2(p^0)^2 \equiv E_\lambda^2(n, p_z) = E_z^2 + E_{T,\lambda}^2(n) \ . \tag{4}$$

The first term in the right side

$$E_z^2 = m_e^2 c^4 + p_z^2 c^2 \tag{5}$$

is the square of the relativistic energy of a free electron moving along the z-axis, whereas the second term defines the square of the relativistic electron transverse energy:

$$E_{T,\lambda}^2(n) = (m_e c^2)\hbar(\frac{eH}{m_e c})(2n + 1 + 2\lambda) \tag{6}$$
$$= (m_e c^2)\hbar\omega_c(2n + 1 + 2\lambda) = 2m_e c^2(\mu_B^e H)(2n + 1 + 2\lambda) \ .$$

The energy spectrum of the Klein-Gordon equation [2, 3] is obtained by omitting the term 2λ in eq.(6). Comparing (3) and (6) the last equation may be expressed as

$$E_{T,\lambda}^2(n) = 2mc^2 \cdot E_{T,\lambda}^{nonrel}(n) \ . \tag{7}$$

Note that for the ground state (with $n = 0$) and spin projection $\lambda = -\frac{1}{2}$, the electron transverse energy is equal to zero in both the nonrelativistic and relativistic approaches:

$$E_{T,\lambda=-\frac{1}{2}}^2(n = 0) \equiv E_{T,\lambda=-\frac{1}{2}}^2(0) = 0 \ , \tag{8}$$

whereas the $n = 0$ level with spin projection $\lambda = +\frac{1}{2}$ has, according to (6), a non-zero transverse energy squared of

$$E_{T,\lambda=+\frac{1}{2}}^2(n = 0) = 2\hbar\omega_c(m_e c^2) = 2\ (2m_e c^2)(\mu_B^e H) \ . \tag{9}$$

So, we realize that *for the ground state with* $\lambda = -\frac{1}{2}$*, the relativistic expression for the total energy of an electron in a static magnetic field* coincides with *the energy of a free electron*

$$E_{\lambda=-\frac{1}{2}}(n=0, p_z) = E_z = \sqrt{m_e^2 c^4 + p_z^2 c^2}\ , \tag{10}$$

which is however not the case for spin projection $\lambda = +\frac{1}{2}$. One also notices from eqs.(6) and (9) that the state with the quantum numbers $n = 0$ and $\lambda = +\frac{1}{2}$ has the same transverse energy as the state with $n = 1$ and $\lambda = -\frac{1}{2}$, i.e.

$$\begin{aligned} E^2_{T,\lambda=-\frac{1}{2}}(n=1) = E^2_{T,\lambda=+\frac{1}{2}}(n=0) = 2\hbar\omega_c(m_e c^2) = \\ = 2\ (2m_e c^2)(\mu_B^e H)\ . \end{aligned} \tag{11}$$

From eq.(6) one derives for the difference $\Delta E^2_{T,\lambda}(n+k|n)$ of the square of two transverse energy levels $E^2_{T,\lambda}$, labeled as $n + k$ ($k = 1, 2, ...$) and n, and identical spin projections λ (i.e. for the non-spinflip case), the following expression

$$\begin{aligned} \Delta E^2_{T,\lambda}(n+k|n) \equiv E^2_{T,\lambda}(n+k) - E^2_{T,\lambda}(n) = 2k\hbar\omega_c(m_e c^2) \\ = 4k(\mu_B^e H)(mc^2) = 2ec\hbar kH = k\Delta E^2_{T,\lambda=-1/2}(1|0)\ . \end{aligned} \tag{12}$$

The energy eigenvalues of eq.(4) in the nonrelativistic limit might be expanded to [6]

$$\begin{aligned} E_\lambda(n, p_z) &= \sqrt{E_z^2 + E^2_{T,\lambda}(n)} = \\ &= \sqrt{m_e^2 c^4 + p_z^2 c^2 + \hbar(\frac{eH}{m_e c})(m_e c^2)(2n + 1 + 2\lambda)} \approx \\ &\approx m_e c^2 + (\frac{p_z^2}{2m_e}) + (\mu_B H)(2n + 1 + 2\lambda) = \\ &= m_e c^2 + E^{nonrel}\ . \end{aligned} \tag{13}$$

This equation clearly reveals the relation between the relativistic energy $E_\lambda(n, p_z)$ and the nonrelativistic energy E^{nonrel}.

3 Comparison of numerical values of transverse energy levels from the Schrödinger and Dirac equations

In the nonrelativistic Schrödinger case the transverse energy is, according to (3), proportional to the strength of the magnetic field H

$$E^{nonrel}_{T,\lambda}(n) \sim (\mu^e_B H) \ , \tag{14}$$

while in the relativistic case the transverse energy is, according to (6), proportional to the square root of H

$$E_{T,\lambda}(n) = \sqrt{2(m_e c^2) E^{nonrel}_{T,\lambda}(n)} \sim \sqrt{2(m_e c^2)(\mu^e_B H)} \ . \tag{15}$$

Obviously, there is a distinct different behavior of both solutions with respect to the magnetic field, which is expected to be more pronounced at larger magnetic field strengths.

For a numerical illustration, let us consider the case for a field of 1 Tesla (1 Tesla=10^4 Gauss, $1erg = 0.624 \cdot 10^{12}$ eV). Utilizing the Bohr magneton of an electron (in Gauss units)

$$\mu^e_B = \frac{e\hbar}{2m_e c} = 0.927\ 10^{-20}\ erg\ Gauss^{-1} = 5.788\ 10^{-9}\ eV\ Gauss^{-1}$$

and eq.(3), the following value for the nonrelativistic transverse energy of the first exited state is obtained

$$E^{nonrel}_{T,\lambda=-\frac{1}{2}}(n=1)_{H=1T} = 2\mu^e_B H = \ \approx 1.158\ 10^{-4}\ eV \ .$$

If this number for $E^{nonrel}_{T,\lambda=-\frac{1}{2}}(n=1)_{H=1T}$ is introduced into (7), the following value for the relativistic transverse energy of an electron being in the same magnetic field with identical quantum numbers n and λ is found

$$E_{T,\lambda=-\frac{1}{2}}(n=1)_{H=1T} \approx 10.87\ eV \ . \tag{16}$$

Comparing the numbers of (16) and (17), one notices that for the magnetic field strength of 1 Tesla the relativistic Dirac equation provides for the first radial excitation a transverse energy *five orders* *of magnitude* higher than the nonrelativistic Schrödinger equation. Technically, this mismatch can be understood from formula (15) because: a) the square root of $\mu_B^e H$ factor enlarges $E_{T,\lambda}$ by about two orders of magnitude and b) the presence of the second multiplication factor, $\sqrt{2(m_e c^2)}$ (with $m_e c^2 = 0.511\ MeV =\ 0.511 \cdot 10^6\ eV$), increases the energy level by additional three orders of magnitude.

The energy of photons emitted by electron transitions from higher $n+k$ to lower n orbits is equal to the difference of the energy of these two orbits. For the nonrelativistic levels (see (3)) with any $n, k \geq 1$ this transition energy is

$$\Delta E_{T,\lambda}^{nonrel}(n+k|n) \equiv E_{T,\lambda}^{nonrel}(n+k) - E_{T,\lambda}^{nonrel}(n) = k(2\mu_B^e H)\ . \quad (17)$$

Its relativistic analog (see (6)) looks as follows

$$\begin{aligned} \Delta E_{T,\lambda}(n+k|n) &\equiv E_{T,\lambda}(n+k) - E_{T,\lambda}(n) = \\ &= \sqrt{m_e c^2 (2\mu_B^e H)}\Delta_\lambda(n+k|n)\ , \end{aligned} \quad (18)$$

where

$$\Delta_\lambda(n+k|n) = \sqrt{2n+2k+1+2\lambda} - \sqrt{2n+1+2\lambda}\ . \quad (19)$$

The ratio of the energies of emitted nonrelativistic photons to the relativistic ones may be written as

$$\frac{\Delta E_{T,\lambda}^{nonrel}(n+k|n)}{\Delta E_{T,\lambda}(n+k|n)} = \sqrt{\frac{(2\mu_B^e H)}{m_e c^2}} F_\lambda(n+k|n)\ , \quad (20)$$

where $F_\lambda(n+k|n)$ is defined as

$$F_\lambda(n+k|n) = \frac{k}{\Delta_\lambda(n+k|n)}\ . \quad (21)$$

Taking into account (16) one finds for $\sqrt{\frac{(2\mu_B^e H)}{m_e c^2}} \approx 1.505\ 10^{-5}$.

Let us consider for simplicity an electron with the polarization $\lambda = -1/2$. In this case

$$\Delta_{\lambda=-1/2}(n+k|n) = \sqrt{2}(\sqrt{n+k} - \sqrt{n}) \quad (22)$$

and, correspondingly (in the case of $\lambda = 1/2$ the expressions under the square root should be enlarged by one unit),

$$F_{\lambda=-1/2}(n+k|n) = \frac{k}{\sqrt{2}(\sqrt{n+k}-\sqrt{n})} \ . \tag{23}$$

This simple formula allows to estimate the influence of the level numbers n and k on the ratio of nonrelativistic and relativistic energies. For n=k=1, it gives $F_{\lambda=-1/2}(n+k|n) \approx 1.707$. For moderate values of $n \leq 10$ one needs to take $k > 100$ in order to increase the value of $F_{\lambda=-1/2}(n+k|n)$ by one order of magnitude. So, one may conclude that for moderate values of n and k higher level transitions have only a small impact on the huge difference (of five orders of magnitude) for the predictions obtained within the nonrelativistic and relativistic approaches. The substantial difference derived for a magnetic field of 1 Tesla, being expected to grow with increasing magnetic field strength, was to our knowledge never discussed in the literature so far.

In this connection it is of interest to recall that the situation with the physical interpretation of the exact solutions of the Dirac equation is not so definite when an electron interacts with a strong electric field. For example, it is well known that the solution for the energy levels of the Dirac equation using the Coulomb potential, $V(r) = -\frac{Xe^2}{r}$, where the charge factor Z defines the strength of the electric field, can be written as

$$E_{n,j}^{Coul.rel} = mc^2(1+\frac{(\alpha Z)^2}{(n-(j+1/2)+\sqrt{(j+1/2)^2-(\alpha Z)^2} \ \ )^2})^{-1/2} \tag{24}$$

with $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ and $n = j+1/2+k = 1,2,...$ the main quantum number, and $j = 1/2$ for $l = 0$ or $j = l \pm 1/2$ if $l \neq 0$. Eq.(25) has however a restricted range of physical validity. For instance, for the smallest value $j = 1/2$ the expression under the square root in the dominator becomes negative and leads to unphysical solutions if $Z > Z_{cr}$, with $Z_{cr} = 137$ as the critical value for the charge factor.

In contrast to the relativistic expression (25), the analogous formula of the bound state energy levels for the Coulomb potential within the Schrödinger equation

$$E_{n,j}^{Coul.nonrel} = -\frac{R\hbar Z^2}{n^2} \ , \tag{25}$$

with the Rydberg constant $R = me^4/2\hbar^3$ and $n = 1, 2, ...$, is valid for all Z values. To resolve experimentally the origin of the discrepancy between the relativistic solution (25) and the nonrelativistic expression (26), in particular for large values of Z, is challenging since stable point-like charges with $Z \geq 137$ are not observed up to now (see, e.g. [4] -[8] for more discussions of this problem in connection with heavy ions case).

Unlike the example of the Coulomb potential, the solution of the Dirac equation (6) for transverse energy levels of electrons within a static magnetic field is valid for any strength of the field H and, thereby, the experimental verification of the predictions (3), respectively, (6) is possible even at very high field strengths. We have not yet found any results of such a study in the literature.

Such a task may be performed by passing an electron beam through a static homogeneous field of about 0.1-10 Tesla. Electrons of up to 100 MeV (which are considered to be relativistic due to the smallness of their mass) may occupy some quasi-stable quantized energy levels $E_{T,\lambda}(n)$. For large life times of such levels, see [12] and [18], transitions of excited electrons to the ground state (10) are restricted during passing through the magnet and registration of emitted photons should be performed sufficiently downstream of the magnet.

Also, "stimulation" of beam electrons by absorption of laser photons inside a magnet leads to "exited" states of electrons which may be followed by emission of light within or behind the magnet depending on the lifetime of the excited states. Details of such an experiment should however be considered as soon as its realization becomes appropriate.

4 Summary

Numerical values for the transverse energy of electrons which propagate in a static homogeneous magnetic field were calculated using the relativistic (Dirac) and the nonrelativistic (Schrödinger) equations. Employing a magnetic field of $0.1 - 10$ Tesla and non-spinflip transitions from orbits with $n = 1$ to $n = 0$, as an example, a difference between the relativistic and nonrelativistic concepts up to five orders of magnitude for the electrons' transverse energy was evaluated. In other words, electrons traversing an external magnetic field of 1 Tesla radiate photons being about 10^5 times more energetic in the relativistic than in the nonrelativistic case, for non-spinflip transitions from $n = 1$ to $n = 0$ orbits.

Finally, we believe that experimental verification of the predictions from either the relativistic or the nonrelativistic equation on quasi-atomic quantized energy levels of electrons traversing a strong static magnetic field would be very desirable. Such measurements can be performed by studying, for example, Compton scattering of laser light with electrons when both beams move parallel along the magnetic field lines. Registration of radiated photons, caused by electron transitions from higher to lower orbits, and the measurement of their energy spectrum should allow to resolve the difference between the relativistic and nonrelativistic predictions and should also be a good test for the way we choose to add the interaction terms to the Dirac equation.

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